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What is School Mathematics? EMS Education Webinar, March 13, 2026

<https://www.youtube.com/watch?v=t7KUhlJ55Sc>

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General disclaimer I will be speaking about *American* school mathematics education in the next hour. This should not be a major problem since we are living in a global village and the American experience is not likely to be unique to America. We should learn from each other.

The question of the title may sound a bit silly. One's first thought might be to turn to Felix Klein for an answer as he had spent so much time pondering over the *double discontinuity* of going from school mathematics to university mathematics, and back. Surprisingly, Klein was so consumed with the mathematical preparation of teachers that all he had to say on this matter was,

If we educate better teachers, then mathematics instruction will improve by itself, as the old consigned form will be filled with a new, revitalized content!

In effect, he anticipated the American “intellectual trickle-down theory”: having good mathematics teachers will trickle down to getting good school mathematics education! Since I am less optimistic, I think we should try to find an answer to this question by ourselves as best we can. There are at least three reasons for us to make this effort.

The first is that the “mathematics” in American schools has been unlearnable for at least seven decades. Are we willing to take this unlearnable “mathematics” as *school mathematics*? And in what way is it unlearnable?

The second reason results from an issue raised by the two recent reforms of **NCTM (1989)** and the **Common Core (2010)**. These reforms hypothesized that “school mathematics” could be “something better”, and they only described *in the barest outline* what this “something better” ought to be. But how do we know that this “something better” could be realized in the real world? Could this be another case of Thomas More’s *Utopia* (1516) — a noble ideal that is not realizable in this imperfect world of ours?

The third and final reason, and perhaps the most obvious one, is the pressing need to replace this unlearnable “mathematics” in mathematics education research with something that is correct and learnable! This unlearnable “mathematics” has ruled over school mathematics education for so long that it is almost the only kind of mathematics known to most researchers in the field.

We begin with the first. In America, the mere mention of *math* would likely trigger extreme reactions. It ranges from hints of an *inferiority complex* — “Our school students have to do better on international tests like PISA or TIMSS” — to outright *mathematical anxiety* — many either flaunt their contempt for mathematics (“I hate math”) or dismiss it as being irrelevant (“I am not a math person”). A 2012 NY Times op-ed even goes so far as to propose the elimination of algebra as a graduation requirement for high school.

Let us see why the mathematics taught in schools for at least the past seven decades has triggered such contradictory reactions. Since this is the “mathematics” in standard American school math textbooks, we will refer to it as **Textbook School Mathematics**, or **TSM**.

Here are some reasons that make TSM unlearnable.

- TSM has few definitions that make sense. Consequently, most students don’t know *precisely* what they are supposed to learn.

- TSM is mainly a collection of cookbook recipes, i.e., *unexplained* and disconnected procedures. The human capacity for retaining such procedures is limited.
- Too often TSM’s lack of precision confuses students.

For the time being, we introduce the concept of *mathematical integrity* to mean roughly the quality of “being precise, clear, logical, coherent, and always supported by reasoning”. This then allows us to summarize TSM’s unlearnability by saying that *TSM has little or no mathematical integrity*.

We will give a precise definition of “mathematical integrity” presently but, first, let us look at a few examples of TSM at work.

(A) In TSM, the four *standard algorithms* for whole numbers are taught as *four disconnected skills to be memorize by rote*. No reasoning. Consequently, what children get from TSM is that mathematics is a subject to be learned by *memorizing facts* and *taking orders from teachers*, but is not about reasoning. Children deserve a better introduction to mathematics than this.

One reason that make the four standard algorithms worth learning is that they reduce all computations with whole numbers — no matter how large — to computations with *single-digit* numbers. This is the scientific principle of **reduction of the complex to the simple** at work. For example, in the case of multiplication, the arduous task of multiplying two large numbers such as 8901234×415678 — which means adding 415678 to itself 8901234 times — is reduced to the (by comparison) simple task of adding up a relatively small number of *single-digit* multiplications. This reduction makes it clear why it is worthwhile to commit the multiplication table to memory once and for all. By not bringing out the reasoning hidden behind the standard algorithms, TSM misses the splendid opportunity of introducing children to this basic scientific principle at an early age and opening their eyes to the fact that

memorizing the multiplication table is as fundamental to doing arithmetic as memorizing the alphabet is to reading and writing English.

TSM makes children memorize the multiplication table without saying a word about why they should do so. This misrepresents mathematics as a subject that relies on brute force memorization for its mastery. Worse, some educators even make a virtue of *not* memorizing this table because, they said, their own failure to do so as a child has never hurt their successful career in mathematics education. We have to lay the blame on TSM for such a sorry spectacle.

(B) TSM *lacks transparency* in explaining almost all the concepts in school mathematics, e.g., it defines a *fraction* as a “part of a whole” or “like a piece of pizza” or a ratio.

To students, what is this “part of a whole”? Is it an “action” that takes some parts from the “whole” (whatever that is) or is it two things — a part *and* a whole? Or, if a fraction is like a pizza, how does a pizza show up in “two-thirds of a mile”? And of course, children have no idea what a “ratio” is when they first approach fractions. Consequently, such a “definition” of a fraction fails utterly to make children feel that a “fraction” *is* a “number” like 8 or 27.

Sometimes, TSM also says a fraction is all three things above, but this only makes a “fraction” three times more incomprehensible.

Without explaining what a “fraction” is, TSM nevertheless has no qualms about drilling students no end on carrying out computations with this mysterious “fraction” object. This is a painful experience to most students because they are forced to do something without knowing what it is all about! This is how the rampant fear of fractions becomes part of the American folklore, a fear that is documented in the popular comic strips **Peanuts** and **Foxtrot**.

Why is so much attention devoted to fractions when discussing elementary school

mathematics? It is because the concept of a fraction is the first major abstraction that children face in mathematics. (For a whole number, they can always look at their fingers, but where can they find $\frac{7}{13}$, for example?) Fractions therefore offer us the first real opportunity to teach children how to handle abstractions in mathematics. Precisely because of the *abstract* nature of the fraction concept, we must first give children a precise description — in other words, a *definition* — of what a fraction is *in terms of what they already know*, and guide them through their first tentative steps of making deductions on the basis of this definition. From this perspective, a definition in terms of “part of a whole” or “ratio” would be entirely inappropriate because the former lacks precision and the latter refers to a concept that children do not know. “Like a piece of pizza” is also inappropriate, but for a different reason: saying a fraction is “like a piece of pizza” is meant to be an analogy, but analogies — while they have a role to play in pedagogical discussions — are not precise statements that can be inserted into a mathematical definition.

(C) Too often, TSM fails to point out structural similarities between certain topics to honor the coherence of mathematics. For example, TSM teaches fractions as “such different numbers” from whole numbers, with the result that the four arithmetic operations on fractions are taught as entirely different skills from those on whole numbers.

This falsifies mathematics (the operations on fractions are conceptually the same as those on whole numbers) and, at the same time, *heightens children’s anxiety about fractions*. This is another reason that makes fractions unlearnable to children.

The same can be said about many other topics. For example, the concepts of length, area, and volume are basically the same (see Chapter 5 of this volume), but TSM usually fails to point out the similarity (or worse, simply does not bother to define any of them precisely), making the learning of these concepts — especially *area* and *volume* — more difficult than necessary.

(D) The notorious high school course in geometry causes disruptions of seismic proportions in the American school mathematics curriculum. I will only mention two of these disruptions.

First, before high school, TSM defines *congruence* of plane figures as “having the same size and the same shape”, and *similarity* as “having the same shape but not necessarily the same size”.

A small digression: As *mathematics*, these definitions are defective beyond repair because it is a *subjective* judgment as to whether or not two figures have the same size or the same shape, but mathematics is about *objective* truths, not subjective judgments. However, we will not be concerned with this particular defect in this talk except to point it out.

In the high school geometry course, however, these definitions of congruence and similarity are abandoned, and congruence and similarity are now defined anew, but *only* for triangles, in terms of the equality of angles and the equality (or proportionality) of sides.

What then does it mean for two plane curves — such as two ellipses — to be congruent or similar in this geometry course? TSM doesn’t say (it feels no need to be accountable). Students coming out of this “rigorous” course therefore become totally confused: what should they believe about the concepts of congruence or similarity? That — for curves in general — they can use the definitions about “size” and “shape”, but that for triangles they must switch to using “equal (or proportional) sides” and “equal angles”? Why? Since TSM does not help them resolve such doubts, mathematical anxiety ensues.

There is in fact a much more serious curricular disruption brought about by the high school geometry course. We have pointed out that TSM seems completely indifferent to the need of any “proof” prior to this course. But in this course, students

are suddenly asked to take the proof of *every* theorem, big or small, with the utmost seriousness. Why? No answer, because TSM is not accountable.

Students know perfectly well that they never got any proof for most of the baffling procedures they were taught such as “invert and multiply” or “negative times negative is positive”. Yet they are now asked to *prove* everything, no matter how obvious, such as:

Given four successive points A , B , C , D on a line as shown below. If $AB = CD$, prove that $AC = BD$.



Even more unsettling is the fact that, in the usual way this geometry course has been taught in American high schools, an acceptable “proof” of this fact consists of *seven* steps in two columns. Moreover, in such a “proof”, the emphasis is not on the *mathematical reasoning* per se but, rather, on students’ ability to closely follow a prescribed protocol.

What students end up learning from this geometry course most of the time is therefore not mathematical reasoning but a fancy way to “fill in the blanks”. Is this what “rigorous proofs” are all about? Students’ sense of betrayal is usually followed by resentment and anxiety.

Summary It does not take great insight to notice the havoc TSM has wrought on public education. Therefore,

TSM has to go.

But if not TSM, *what kind of mathematics do we envision for our schools?* Once again, this harks back to our original question: What is school mathematics?

In America, there seems to have been little serious attempt to answer this basic question until the 1950s. Between 1950 and now, there have been three major reforms in school mathematics education: the New Math of 1957, the NCTM reform of 1989, and the Common Core reform of 2010. Each of these reforms tried to offer its version of a “better” alternative to TSM. It is in the process of describing these “better” alternatives that we’ll come fact-to-face with our second reason for raising the question in the title.

A major caveat. While the NCTM and the Common Core reforms will be linked together throughout this article as a result of their similarity when looked at from a certain perspective, we are by no means saying that these two reforms are identical. There is a noticeable difference in the mathematical integrity of their individual standards, for example.

For the purpose of understanding these three reforms, the best way is to put them in the context of the two basic demands on school mathematics:

- (*i*) The *substance* of the mathematics must be consonant with the requirements of mathematical integrity.
- (*ii*) The *level of presentation* of the mathematics must be suitable for consumption by school students.

In terms of these two demands, TSM is a version of school mathematics that ignores (*i*) but ruthlessly pursues (*ii*) in the name of accessibility by stripping the mathematics down to only its procedures — which unfortunately also banishes reasoning in the process.

TSM ruled the American schools when, in October of 1957, Russia’s launching of Sputnik I shook the nation to its core. Washington DC responded by launching a school mathematics education reform: the **New Math**.

Mathematicians were put in charge of the New Math reform and, predictably, they concentrated on **(i)** at the expense of **(ii)**. Since the mathematics used by mathematicians — to be called **university mathematics** — had achieved complete transparency and accountability by 1930 (one thinks of David Hilbert, Emil Artin, Emmy Noether, and B. L. van der Waerden), it was natural for the New Math to approach school mathematics by simply replacing TSM with the elementary portion of university mathematics. So elementary students had to learn about sets, numbers in bases other than 10, the difference between a “numeral” and a “number”, modular arithmetic, and symbolic logic.

To be fair, the mathematicians did make an effort to tone down the abstractions and the formalism. To their credit, they also saw the need to provide guidance for textbook publishers and teachers’ professional development by producing a complete set of textbooks for grades K to 12. Nevertheless, the change in the mathematical landscape was too drastic and the new regimen did not sit well with students, their parents, *and* especially their teachers. The New Math ultimately failed to take root in America and, as a result of the resistance from all sides, it had all but vanished by the early 1970s.

Recall the two basic demands on school mathematics, **(i)** and **(ii)** above. We have just seen

the overemphasis of **(ii)** over **(i)** by TSM, and

the overemphasis of **(i)** over **(ii)** by the New Math.

We have also seen that both kinds of overemphasis are self-defeating. A new approach to school mathematics is called for.

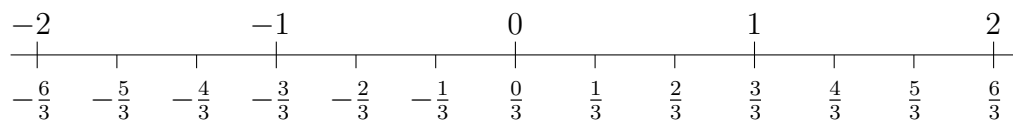
Here is the new approach we propose: create a new version of school mathematics by the use of *mathematical engineering*.

In other words, one *customizes*, or *engineers* university mathematics — **if possible**

— to render it consumable by school students and teachers, in the same way that, e.g., electrical engineers customize Maxwell’s electromagnetic theory to create tools that are consumable by ordinary people, e.g., cell phones, computers, lighting fixtures, etc., or civil engineers customize Newtonian mechanics to build freeways, bridges, and skyscrapers.

A lot can be said about the engineering process of *customizing university mathematics* for the consumption of school students. Here is a very brief description of one example.

In university mathematics, the rational numbers $\mathbf{Q}$ (the positive *and negative* fractions) are obtained from the integers by taking the latter’s quotient field. This is too abstract for school mathematics. The *engineering decision* is to start with the integers defined as an infinite sequence of equidistant points on a line L and add $\mathbf{Q}$ to L as points of equal subdivision between consecutive integers. For example, here are some of the rational numbers with denominator 3:



See Chapter 2 of this volume.

A fundamental tenet of engineering is that any kind of engineering *has to be faithful to the abstract theory* or else disaster awaits (bridges will collapse, whole cities will go dark, etc.). Mathematical engineering is no exception: it must be consonant with the requirements of mathematical integrity mentioned in **(i)**. But now, we must spell out what *exactly* “mathematical integrity” means for this statement to make sense! We say a version of school mathematics *has mathematical integrity* if it satisfies the following five fundamental principles.

1. Every concept is **precisely defined** to let students know what they are learning.

2. Every mathematical claim is supported by **logical reasoning** (i.e., *proved*) to show why it is true.
3. The language used is **precise** so as to avoid any misunderstanding of the intended message.
4. The diverse concepts and skills are shown to be parts of a **coherent** whole.
5. The **purpose** for learning each concept or skill is consistently made clear.

But TSM consistently violates all five:

For **1**, e.g., “fraction” has no definition.

For **2**, e.g., why is the long division algorithm correct?

For **3**, e.g., can “same size and same shape” be used as a definition of “congruence”?

For **4**, e.g., why does the high school course on geometry stick out like a sore thumb from the school mathematics curriculum?

For **5**, e.g., why bother to study the four standard algorithms for whole numbers?

We call a version of school mathematics

PBSM,

Principles-Based School Mathematics, if it satisfies both demands **(i)** and **(ii)**.

Key point: *We don’t as yet know whether or not PBSM exists as a coherent body of knowledge*, or if it exists, whether PBSM can materialize in real K-12 classrooms. We can amplify slightly on the former. While in theory, mathematical engineering

can be applied to any topic in the mathematics of K-12, it is far from clear that it can be done successfully for *all* the topics in the K-12 curriculum. At least, there is no record of such success. However, we will see presently that PBSM was implicitly assumed to be accessible by the NCTM (1989) and the Common Core (2010) reforms, and this fact turns out to have a bearing on the outcomes of the reforms.

The concept of PBSM brings us to the second of the three reasons that prompted us to ask “What is school mathematics?”. The **NCTM reform** of 1989 and the **Common Core reform** of 2010, *in essence*, wanted PBSM to replace TSM. In other words, both reforms wanted “school mathematics” to be PBSM. Let it be known once and for all that neither reform used the terminology of “TSM” or “PBSM” but, in retrospect, we feel that a discussion of these reforms from the perspective of “TSM” and “PBSM” is especially productive.

Both are *standards-based reforms*, i.e., they issued a set of *standards* that dictate what topics to cover in each grade and in what order they should be taught. The hope was that, by following these standards, the mathematics in schools would emphasize

conceptual understanding, mathematical problem solving, coherence, and reasoning and proof.

Moreover, the Common Core reform also called for *procedural fluency* and *clear definitions*. In any case, we recognize that these qualities are basically consistent with what PBSM hopes to achieve. Both reforms implicitly assumed that PBSM was already “out there”, and all they had to do was to use a set of standards to nudge the schools a bit and PBSM would be realized in all the math classrooms.

Unfortunately, those standards merely indicate *in the barest outline* what to teach and how to teach it, but the details were missing. This is akin to issuing an edict that a building of prescribed dimensions should be erected in some inaccessible mountainous region without also giving detailed instructions on how to get it done. What makes

this analogy especially relevant is the fact that in an environment ruled by TSM, the reform’s clarion call for “conceptual understanding”, “coherence”, etc., might as well fall on deaf ears. Few, if any teachers, school district administrators, and mathematics educators, had any idea what these phrases meant. It therefore came to pass that when well-intentioned educators and teachers tried to implement the standards, they ran into a blank wall. They needed *mathematical* help!

The reforms also did not attend to the most basic needs for a successful implementation of the standards, e.g.,

- How to provide teachers with the content knowledge of PBSM.
- How to replace TSM-textbooks with PBSM-textbooks.
- How to get assessment-items writers who also know PBSM to create new assessment-items that are consonant with PBSM.

It follows that if there was to be any hope of success for either reform, the first step would have to be

making available a *complete and detailed* exposition of PBSM

to help teachers and administrators come to some understanding of the standards. The hard work of implementation would have to be the next step.

Given professional mathematicians’ facility with developing a topic *ab initio*, many of them could have written such an account of PBSM. Of course, they would have needed the willingness and the time to look at lots of TSM textbooks, as well as the humility to learn about what was happening in mathematics classrooms in schools.

Apparently nobody did, so the reforms basically didn’t stand a chance.

I took on the challenge in 2000, without anticipating that the Common Core reform would be introduced in 2010. By 2020, six volumes covering PBSM from K to 12 totaling 2402 pages have appeared in print. Each volume is essentially the lecture

notes of a summer institute or a one-semester university course for the professional development of teachers in a particular grade band.

- Understanding Numbers in Elementary School Mathematics (2011)
- Teaching School Mathematics: Pre-Algebra (2016)
- Teaching School Mathematics: Algebra (2016)
- Rational Numbers to Linear Equations (2020)
- Algebra and Geometry (2020)
- Pre-Calculus, Calculus, and Beyond (2020)

They are all published by the American Mathematical Society. The first volume is on the mathematics of elementary school. The next two volumes are on the mathematics of middle school. The last three volumes are on the mathematics of high school.

The main virtue that these six volumes can claim is that they furnish the first existence proof of PBSM in grades K-12.

As emphasized earlier, having such a complete and detailed exposition of PBSM is only the beginning of our ongoing battle to win a better school mathematics education for the next generation. What lies ahead is the arduous task of implementation: how to get PBSM into school classrooms. It will be extremely time-consuming and very expensive. Please see the last section of this article.

Beyond this, I hope these six volumes will stir up some debate. For example, do they actually have mathematical integrity? Is the mathematical level pitched too high for school students of 2026 and beyond? Can they survive the inevitable intrusion of AI in K-12 education? More importantly, I hope soon there will be other competing versions of PBSM.

The competition will eventually usher in an era in which PBSM becomes the default version of school mathematics. When that happens, we will have made some

real progress in school mathematics education.

Recall that we have three reasons for questioning what *school mathematics* is. We now come to the third reason, the one about research in school mathematics education.

The oppressive dominance of TSM on education research in America cannot be overstated. After all, TSM has been almost the only version of school mathematics ever discussed in American schools of education. TSM being what it is — intellectually execrable — research about TSM becomes an exercise in *refuse-reclamation*, i.e., it is as if the education researchers are saying:

We know TSM is no good, but let us see what we can do to make certain topics more palatable to students by infusing them with a bit of conceptual understanding and reasoning.

However, *students should be fed healthy food, not reclaimed scraps from garbage piles*. Such scholarly effort would have been better spent on devising the best pedagogy to help students learn mathematics with mathematical integrity. Instead, it has been wasted on making TSM less toxic.

There is an additional reason for rejecting TSM in education research. When TSM is dignified by scholarly attention, it gains legitimacy because teachers and students get the impression that

TSM is an integral — though unfortunate — part of life, and there is nothing they can do about it.

In this light, the NCTM reform of 1989 has the unquestioned merit of being a revolt initiated within the inner circle of mathematics education. May this revolt reach the desired conclusion by laying TSM to rest once and for all!

Finally, and it may be anticlimactic by now, but my answer to the question in the title of this talk is: *“School Mathematics” should be PBSM*.

Let us hope!

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